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Cubic Interpolatory Splines with Nonuniform Meshes

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1. INTRODUCTION

Cubic spline interpolation problems of matching the spline at one intermediate point and spline with multiple knots at two intermediate points between the successive mesh points have been studied in [6]. The former of these problems has been answered only for the case of uniform meshes. For this case, further studies in the direction of the result proved in [6] have been made in [2–4]. The object of the present paper is to study the existence, uniqueness, and convergence properties of cubic spline interpolant matching at one intermediate point between the successive mesh points, which are not necessarily equispaced. Considering a geometric mesh we shall show that nonuniform meshes permit a wider choice for the points of interpolation than those possible for the case of uniform meshes. Interesting studies of general cardinal spline interpolation on a geometric mesh have been made by Micchelli [5, p. 241].

2. EXISTENCE AND UNIQUENESS

Let a mesh on [0, 1] be given by

$$P: \{0 = x_0 < x_1 < \cdots < x_n = 1\}$$

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with $h_i = x_i - x_{i-1}$, $\bar{h} = \max_i h_i$, and $h = \min_i h_i$ for $i = 1, 2, \ldots, n$. Let π_k denote the set of all algebraic polynomials of degree not greater than k. For a function s defined over P we denote the restriction of s over $[x_{i-1}, x_i]$ by s_i. The class $S(3, P)$ of cubic splines defined over P is given by

$$S(3, P) = \{s: s \in C^2, s_i \in \pi_3 \text{ for } i = 1, 2, \ldots, n\}.$$

Writing $t_i = x_{i-1} + \theta h_i$ with $0 \leqslant \theta \leqslant 1$ and considering a given function f we introduce the interpolatory conditions

$$s(t_i) = f(t_i), \qquad i = 0, 1, \ldots, n, \tag{2.1}$$

where $t_0 = 1$ (or 0) if θ (or $(1-\theta)$) lies in a subinterval of $[0, 1]$ which includes 0. We pose the following.

PROBLEM A. *Suppose f' exists at 0 and 1 and $s'(j) = f'(j)$, $j = 0, 1$. Then does there exist a spline interpolant $s \in S(3, P)$ of f which satisfies (2.1)?*

It may be observed that the existence and uniqueness of the complete cubic spline interpolant of f matching at the mesh points answers Problem A when $\theta = 0$ (see [1]).

In order to answer Problem A for other choices of θ, we set $s''(x_i) = M_i$ and observe that for the interval $[x_{i-1}, x_i]$,

$$6h_i s(x) = (x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i + 6h_i(x - t_i) c_i + 6h_i d_i, \tag{2.2}$$

where c_i and d_i are appropriate constants. For any sequence $<\alpha_n>$ we write $\delta\alpha_n$ for $(1-\theta)\alpha_n + \theta\alpha_{n+1}$ and notice that since $s \in C^2$,

$$2\Delta c_i = M_i(h_i + h_{i+1}); \qquad \tfrac{1}{6} M_i \Delta h_i^2 = -\Delta d_i + \delta(h_i c_i), \tag{2.3}$$

where Δ is the usual forward difference operator.

For any function g of θ, h_{i-1}, h_i, and h_{i+1}, we write for convenience g^* for the function obtained from g by interchanging θ with $\theta^* = (1-\theta)$ and h_{i-1} with h_{i+1}.

Eliminating c_i, d_i between Eqs. (2.1)–(2.3) we get

$$\begin{aligned} &R_i M_{i+1} + T_i M_i + T_i^* M_{i-1} + R_i^* M_{i-2} \\ &\quad = 6[(\delta h_{i-1}) \Delta f(t_i) - (\delta h_i) \Delta f(t_{i-1})], \qquad 1 < i < n, \end{aligned} \tag{2.4}$$

where $R_i = \theta^3 h_{i+1}^2 \, \delta h_{i-1}$ and

$$\begin{aligned} T_i = {} & [(3-\theta)\theta^2 h_{i+1}^2 + (1-\theta^2) h_i^2] \, \delta h_{i-1} + \theta^2 h_i^2 \, [\theta^{*2} h_{i-1} + (3-\theta^2) h_{i+1}] \\ & + 3\theta \, \theta^* h_{i-1} h_i h_{i+1}. \end{aligned}$$

Since $0 \leqslant \theta \leqslant 1$, it may be directly seen that in Eq. (2.4), R_i, R_i^*, T_i, and T_i^* are all nonnegative.

The system of $(n-2)$ equations given by (2.4) leaves three degrees of freedom which are supplemented by the boundary conditions given by Problem A. We are thus set to prove the following.

THEOREM 2.1. *There exists a unique s in $S(3, P)$ satisfying the requirements of Problem* A *if* (i) $0 \leqslant \theta \leqslant 0.44$ *and* $\langle h_i \rangle_{i=1}^n$ *is nonincreasing or if* (ii) $0.56 \leqslant \theta \leqslant 1$ *and* $\langle h_i \rangle_{i=1}^n$ *is nondecreasing.*

3. PROOF OF THEOREM 2.1

We set

$$J(\theta) = 1 - 6\theta^2 + 2\theta^3, \qquad r_i = \theta^{*2}(1+2\theta)h_{i-1}^2 - 3\theta^2 h_i^2, \tag{3.1}$$

and first consider the case (i). Using the interpolatory requirements of Problem A, Eq. (2.2), and the assumption that $s \in C^2$, we obtain

$$M_0 l_0 + M_1 l_1 + M_2 l_2 = 6h_1 [f(t_2) - f(t_1) - f'(x_0)\,\delta h_1], \tag{3.2}$$

$$M_{n-2}\bar{l}_2 + M_{n-1}\bar{l}_1 + M_n \bar{l}_0 = 6h_n [f'(x_n)\,\delta h_{n-1} - f(t_n) + f(t_{n-1})], \tag{3.3}$$

$$M_{n-1}\theta^{*3} h_n^3 + M_n \theta^{*2}(2+\theta)\, h_n^3 = 6h_n [\theta^* h_n f'(x_n) - f(x_n) + f(t_n)], \tag{3.4}$$

where $l_0 = 3h_1^2\,\delta h_1 - \theta^{*3} h_1^3$; $l_2 = \theta^3 h_1 h_2^2$; $l_1 = l_0 - l_2 - h_1 r_2$ and $\bar{l}_0$, $\bar{l}_1$, and $\bar{l}_2$ are respectively obtained from l_0, l_1, and l_2 by interchanging θ with θ^*, h_1 with h_n, and h_2 with h_{n-1}. Collecting Eqs. (3.2), (2.4) with $i = 2, 3, ..., n-1$, along with (3.3)–(3.4) in that order, we may write them as

$$C_1(M_i) = (F_i), \tag{3.5}$$

where C_1 is the coefficient matrix, (M_i) is the single column matrix, and (F_i) is the single column matrix of the values on the right-hand side of the system of equations under consideration.

We first notice that in C_1 the excess of the diagonal element over the sum of other elements in ith row for $1 < i < n$ is

$$r_{i+1}\,\delta h_{i-1} + (r_i + 2\theta^3 h_i^2)\,\delta h_i = A_i, \tag{3.6}$$

say. Using (i) we observe that $J(\theta)$ is a decreasing function of θ for relevant values of θ so that $J(\theta) \geqslant J(0.44) > 0$ and we have

$$r_i \geqslant J(\theta)\, h_n^2, \tag{3.7}$$

$$A_i \geqslant 2(J(\theta) + \theta^3)\, h_n^2\,\delta h_{n-1} \geqslant J(\theta)\, h^3. \tag{3.8}$$

Again using (i) and (3.6), we see that the elements of C_1 in ith row for $i = 1, n, n+1$ are nonnegative and the excess of the diagonal element over the sum of the other elements in each of these rows is not less than $J(\theta)\, h^3$. Thus C_1 is a diagonally dominant matrix so that C_1^{-1} exists and its row-max norm is

$$\|C_1^{-1}\| \leqslant (J(\theta)\, h^3)^{-1}. \tag{3.9}$$

This proves Theorem 2.1 in the case (i).

In the other case in which (ii) holds, we see that the boundary conditions of Problem A yield (3.2)–(3.3) and the following in place of (3.4):

$$M_0 \theta^2 (3-\theta)\, h_1^3 + M_1 \theta^3\, h_1^3 = 6h_1 \left[f(t_1) - f(x_0) - \theta h_1 f'(x_0) \right]. \tag{3.10}$$

Now rearranging Eqs. (3.10), (3.2), (2.4) with $i = 2, 3, ..., n-1$ and (3.3) in that order, we may write them as

$$C_2(M_i) = (F_i^*), \tag{3.11}$$

where C_2 is the coefficient matrix and (F_i^*) is the single column matrix of the values on the right-hand side of the system of equations under consideration.

Again we observe that in C_2 the excess of the diagonal element over the sum of the other elements in ith row for $2 < i \leqslant n$ is A_i^*. But in view of the condition (ii) we see that $J(\theta^*)$ is an increasing function of θ so that $J(\theta^*) \geqslant J(0.56) > 0$ and we have

$$r_i^* \geqslant J(\theta^*) h_1^2, \tag{3.12}$$

$$A_i^* \geqslant 2(J(\theta^*) + \theta^{*3}) h_1^2\, \delta h_1 \geqslant J(\theta^*) h^3. \tag{3.13}$$

We also see that the difference of the diagonal element in the ith row for $i = 1, 2, n+1$ over the sum of the other elements in each row is not less than $J(\theta^*) h^3$. Thus C_2 is invertible and the row-max norm of C_2^{-1} is

$$\|C_2^{-1}\| \leqslant (J(\theta^*) h^3)^{-1}. \tag{3.14}$$

This completes the proof of Theorem 2.1.

4. Error Bounds

In this section we shall obtain bounds for the function $e = s - f$ where s is the complete cubic spline interpolant of f in $S(3, P)$. Given any function g we write for convenience $g(x_i) = g_i$ and $w(g, h)$ for the modulus of continuity of g.

We first consider the spline interpolant s of f under the condition (i) of Theorem 2.1. Writing Eq. (3.5) as

$$C_1(e_i^{(2)}) = F_i - C_1(f_i'') = (D_i), \tag{4.1}$$

we first estimate $|D_i|$.

Setting $K_i(j, k) = 3(\delta h_{i-1})^j (\delta h_i)^k$, we observe that

$$T_i = K_i(1, 2) + K_i(2, 1) - T_i^* - R_i^* - R_i. \tag{4.2}$$

Now applying Taylor's theorem, we see that for $i = 2, 3, \cdots, n-1$,

$$F_i = K_i(1, 2) f''(y_{i+1}) + K_i(2, 1) f''(y_i),$$

where $y_i \in (x_{i-2}, x_i)$ and, therefore,

$$\begin{aligned} D_i &= F_i - R_i f''_{i+1} - T_i f''_i - T_i^* f''_{i-1} - R_i^* f''_{i-2} && (4.3)\\ &= K_i(1, 2)(f''(y_{i+1}) - f''_i) + K_i(2, 1)(f''(y_i) - f''_i) \\ &\quad - R_i \Delta f''_i + T_i^* \Delta f''_{i-1} + R_i^*(f''_i - f''_{i-2}). && (4.4) \end{aligned}$$

Thus, in view of (4.2), we have for $1 < i < n$

$$|D_i| \leqslant (2K_i(1, 2) + 3K_i(2, 1) - T_i + R_i^*)\, w(f'', \bar{h}). \tag{4.5}$$

But under the condition (i) of Theorem 2.1 we have

$$(R_i^* + T_i) \leqslant 2(1 + 3\theta^2 - 2\theta^3)\bar{h}^3 \leqslant 3\bar{h}^3,$$

so that for $1 < i < n$

$$|D_i| \leqslant 19\bar{h}^3\, w(f'', \bar{h}). \tag{4.6}$$

Observing that $l_0 + l_1 + l_2 = h_1 K_1(0, 2) + 2\theta h_1^2 K_1(0, 1)$ and using Taylor's theorem, we notice that for some $z_1 \in (x_0, x_1)$

$$\begin{aligned} |D_1| = |h_1 K_1(0, 2)(f''(y_2) - f''_1) + 2\theta h_1^2 K_1(0, 1)(f''(z_1) - f''_1) \\ + l_0 \Delta f''_0 - l_2 \Delta f''_1| \leqslant 15\bar{h}^3 w(f'', \bar{h}). \end{aligned} \tag{4.7}$$

By a similar argument we notice that

$$|D_i| \leqslant 15\bar{h}^3 w(f'', \bar{h}) \qquad \text{for } i = n, n+1. \tag{4.8}$$

Combining (3.9) with (4.6)–(4.8), we observe that

$$\|(e_i'')\| \leqslant (G(\theta) - 1)\, w(f'', \bar{h}) \tag{4.9}$$

where $G(\theta)-1=19(\bar{h}/h)^3/J(\theta)$. Now using the standard arguments (see [6, p. 246]) it follows that

$$\|e^{(2)}\| \leqslant G(\theta)w(f'', \bar{h}) \tag{4.10}$$

and

$$\|e^{(r)}\| \leqslant 2(\bar{h})^{2-r}\, G(\theta)w(f'', \bar{h}) \qquad \text{for } r=0, 1. \tag{4.11}$$

This proves the following when the condition (i) holds.

THEOREM 4.1. *Suppose that $f\in C^2$ and s is the complete cubic spline interpolant of f of Theorem* 2.1. *Then for $r=0, 1, 2$,*

$$\|(s-f)^r\| \leqslant 2K(\theta)(\bar{h})^{2-r}w(f'', \bar{h}) \tag{4.12}$$

where $K(\theta)$ is some positive function of θ.

Following closely the proof of Theorem 4.1 for the case (i), we obtain (4.12) for the case (ii) with $K(\theta)=G(\theta^*)$.

5. PERIODIC SPLINES

The boundary conditions

$$s^r(0)=s^r(1) \qquad \text{for } r=0, 1, 2, \tag{5.1}$$

alongs with (2.1), define the periodic spline interpolant of f at the points t_i. It is worth noticing that the boundary condition (5.1) yields equations which along with (2.4) have a coefficient matrix without diagonal dominant property for $0<\theta<1$ and we are, therefore, unable to conclude the uniqueness of the periodic cubic spline interpolant of f at points other than the mesh points. However, taking $f\in C^2$ and assuming that f and s are 1-periodic so that *a fortiori* (5.1) holds, Meir and Sharma [6] have studied interpolation by such a cubic periodic spline for the case of uniform mesh. In order to study a similar problem for nonuniform meshes we introduce the following.

Considering an extension of a given strictly increasing sequence $y=\langle y_i\rangle_{i=0}^{n}$ to a strictly increasing sequence $\langle y_i\rangle_{i=0}^{n+2}$, we say that a function f is in class E_y if $f(y_j)=f(y_{j+n})$ for $j=0, 1, 2$. Cubic splines s satisfying the conditions

$$s\in E_t \qquad \text{and} \qquad s''\in E_x \tag{5.2}$$

define the class $S_1(3, P)$ of extended periodic splines. We shall now deduce the following from the proof of Theorem 2.1.

THEOREM 5.1. *Suppose* $\langle f(t_i)\rangle_{i=1}^{n+1}$ *are given functional values. If* (i) $0 \leqslant \theta \leqslant 0.44$ *and* $\langle h_i\rangle_{i=1}^{n+1}$ *is nonincreasing or if* (ii) $0.56 \leqslant \theta \leqslant 1$ *and* $\langle h_i\rangle_{i=1}^{n+1}$ *is nondecreasing then there exists a unique spline* s *in* $S_1(3, P)$ *which satisfies the interpolatory condition* (2.1). *Further, if* $f \in C^2$ *and* $f'' \in E_x$ *then*

$$\|e^{(r)}\| \leqslant 2(\bar{h})^{2-r} y(\theta) w(f'', \bar{h}) \qquad \text{for } r = 0, 1, 2, \tag{5.3}$$

where $y(\theta) - 1 = 19(\bar{h}/h)^3/J(\theta)$.

Proof of the Theorem 5.1. Since $s \in E_t$ it follows from the interpolatory condition (2.1) that $f \in E_t$. Thus considering the extensions of $\langle x_i\rangle$ we get Eqs. (2.4) for $i = 2, 3, \ldots, n+1$ which are sufficient to determine M_i's. The proof of the existence part of Theorem 5.1 follows from the diagonal dominant property for Eqs. (2.4) which has already been demonstrated in the proof of Theorem 2.1. The proof of the remaining part of the Theorem for case (i) essentially follows from (3.9) and (4.6). In the other case the proof is similar.

It may be observed that if we assume the mesh points to be uniform, then our Theorem 5.1 proves the existence and uniqueness of the cubic spline interpolant considered in [6].

6. INTERPOLATION ON A GEOMETRIC MESH

We shall show in this section that equispacing of mesh points produces certain limitations on the choice of the points of interpolation which could be avoided otherwise. To support this we derive the following which shows that the point of interpolation could be chosen anywhere between the successive mesh points whereas in the case of uniform meshes this is not possible (see [3, 6]).

COROLLARY 6.1. *If* $h_{i-1}/h_i = 1.12$ (or $(1.12)^{-1}$) *for all* i *then for* $0 \leqslant \theta \leqslant 1/2$ (*or* $1/2 \leqslant \theta \leqslant 1$), *there exists a unique spline* s *in* $S_1(3, P)$ *which satisfies the interpolatory condition* (2.1).

Proof of the Corollary. Writing $v = 1.12$ it follows from (3.6) that the excess of the coefficient of M_{i-1} over the sum of the coefficients of M_{i-2}, M_i, and M_{i+1} is

$$U(\theta) = [(1+v)(\theta^{*2}(1+2\theta)v^2 - 3\theta^3) + 2v\theta^3](\theta^* v + \theta)v^{-2} h_i^3 \tag{6.1}$$

which is positive if $0 \leqslant \theta \leqslant 1/2$. Next we observe that in the other case in which $h_i/h_{i-1} = v$, the excess of the coefficient of M_i over the sum of the coefficients of M_{i-1}, M_{i-2}, and M_{i+1} is $U(\theta^*)$ which is clearly positive for $1/2 \leqslant \theta \leqslant 1$.

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